

# Chain conditions over polynomial and power series rings

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- Notions

- Examples and results

- Main Thm, Proof Technique

# Notions and results

Ascending chain condition:

Let  $P$  be a Poset.  $P$  is ACC (DCC) if <sup>Descending</sup>

$a_1 < a_2 < a_3 < \dots$  Terminates ( $a_n = a_{n+1} = \dots$ )  
 $a_1 > a_2 > a_3 > \dots$

$P$ : ACC  $\iff$  Every nonempty subset of  $P$  has  
DCC maximal element.  
minimal

Noetherian ring:  $I_i \triangleleft R$

- ①  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_k \subseteq \dots$  Terminates.
- ② Every set ideals has maximal element.
- ③ Every ideal is finitely-generated.

Example:  $\mathbb{R}$ ,  $\mathbb{Q}$ ,  $\mathbb{F}$ ,  $\mathbb{Z}/m\mathbb{Z}$ , any finite ring.

Artinian rings:  $I_i \triangleleft R$

$I_1 \supseteq I_2 \supseteq \dots \supseteq I_k \supseteq \dots$  Terminates

Example.  $F$ ,  $\mathbb{Z}/m\mathbb{Z}$ , finite rings.

Hilbert basis theorem.

$R$ : Noetherian  $\longrightarrow R[x], R[[x]]$ : Noetherian

(Main focus of this talk

Krull dimension.

supremum length of all chains of prime ideals in  $R$ .

$$\underbrace{P_1 \subseteq P_2 \subseteq \dots \subseteq P_n}_{\text{length } n}$$

$R$ : Noetherian  $\longrightarrow$  Every prime has length  $< \infty$

(Not necessarily for  $R$   
Nagata example:  $k[x_1, x_2, \dots]$ )

## Ascending chain condition on principal ideals .

$$\begin{array}{l} a_1R \subseteq a_2R \subseteq a_3R \subseteq \dots \\ Ra_1 \subseteq Ra_2 \subseteq Ra_3 \subseteq \dots \end{array} \quad \begin{array}{l} \text{Terminates} \\ \text{Terminates} \end{array} \quad \begin{array}{l} \rightsquigarrow \text{AccPL} \\ \rightsquigarrow \text{AccPR} \end{array}$$

if  $R$  is commutative,  $\text{AccPL} = \text{AccPR} = \text{AccP}$ .

Note:  $\text{DCCPL}(R)$  is also important, for example

$\text{DCCPL} =$  right perfect ring (Bass 1960)

let  $R$ -module + flat = projective

### Skew polynomial ring :

$$\alpha \in \text{End}(R),$$

$$R[x, \alpha] = \{ f : \text{polynomial in } R \} \text{ with } xa = \alpha(a)x \quad \forall a \in R.$$

$$\text{Example: } R = \mathbb{F}_{11}[t], \quad \alpha(t) = -t, \quad \alpha(1) = 1.$$

$$\begin{aligned} (3+tx)(3t+5x) &= 9t + tx3t + 15x + tx5x \\ &= 9t - 3t^2x + 4x + 5tx^2 \\ &= 9t + (4-3t^2)x + 5tx^2 \end{aligned}$$

Skew power series  $R[[x, \alpha]]$  ring is also defined similarly.

Questions:

① Do we have Hilbert basis theorem for  $R[x, \alpha]$ ,  $R[[x, \alpha]]$ ? If not, what conditions on  $R, \alpha$  should we have to get the Hilbert basis theorem?

② Do we have a similar theorem like the Hilbert basis theorem for ACCP instead of Noetherian (ACC)?  
How about Hilbert basis theorem for ACCP on  $R[x, \alpha]$  or  $R[[x, \alpha]]$ ?

Atomic ring.

Every  $a \in R$  can be written as a finite product of irreducible elements (atoms).

Example: Noetherian domain, fields, UFD, ...

Grams (1974):  $R$ : Commutative

$R$ : Atomic, domain  $\implies R[x_1, x_2, \dots]$ : Atomic, domain

Grams (1974):  $R$ : Commutative

$R$ : Domain

$\bar{R}$ : Integral closure of  $R$ : ACCP  $\left. \vphantom{\bar{R}} \right\} \implies R$ : ACCP.

Are atomic rings ACCP?

Zaks (1980):

$F = K[z^2, z^1, z^{-1}, z^{-2}, \dots]$

$K = k(x_1, x_2, \dots)$

$k$ : field

Then  $R = [x_1, \dots, \frac{z}{x_1}, x_2^{-1}, \frac{z^{-1}}{x_3}, \frac{z^{-2}}{x_4}, \dots, \frac{z^2}{z}, \frac{z^1}{z}, \frac{z^{-1}}{z}, \dots]$

is atomic, but does not satisfy ACCP.

So the answer is No 😞.

Heinzer, Lantz (1994):  $K = k[x_1, x_2, \dots]$

$k$ : field

Let  $I = \{x_n(x_n - x_{n-1}), n \geq 2\} \subseteq K$  and  $S = \frac{K}{I}$ .

Define  $R$  is  $S_{\langle \bar{x}_1, \bar{x}_2, \dots \rangle}$  be localization.

Then  $R$  satisfies ACCP, but  $R[y]$  does not satisfy ACCP:

$(\bar{x}_1 y + 1)R[y] \subseteq (\bar{x}_2 y + 1)R[y] \subseteq \dots$  does not terminate.

Do we have a similar theorem like the Hilbert basis theorem for ACCP

instead of Noetherian (ACC)? **No** 😞

First good news.

$R$ : Noetherian  $\left\{ \begin{array}{l} \longrightarrow R[x, \alpha] \text{ is noetherian.} \\ \alpha: 1-1 \end{array} \right.$

Skew Hilbert basis theorem?

Question: If  $R[x, \alpha]$  is noetherian, can we say  $\alpha$  is 1-1? VERY HARD

Reduced rings: for every  $a^2=0$ ,  $a=0$ .

So  $xy=0 \Rightarrow yx=0$ .

ACC on ann.

$\text{Ann}(I_1) \subseteq \text{Ann}(I_2) \subseteq \text{Ann}(I_2) \subseteq \dots$  terminates.

Frohn (2002):  $R$ : commutative

$R$ : ACCP

$R$ : ACC on annihilators

$R$ : Reduced

$\left\{ \begin{array}{l} \implies R[x] : \text{ACCP} \end{array} \right.$

Note: Frohn has counterexample for

$R$ : ACCP

$R$ : Reduced

$\left\{ \begin{array}{l} \not\implies R[x] : \text{ACCP.} \end{array} \right.$

# Generalized power series (Monoid ring):

$R$ : ring

$(S, \leq)$ : strictly ordered monoid ( $a > b \Rightarrow ac > bc$  or  $ac < bc$ ).

$$R[[S]] = \{f: S \rightarrow R\}, \text{supp}(f) = \{s \in S; f(s) \neq 0\}$$

all  $a_i \neq 0$  in powerseries

$\text{Supp}(f)$ : artinian  $s_1 > s_2 > s_3 > \dots$  terminates

narrow  $\rightarrow$  each pairwise incomparable subset has finitely element.

and

$$(fg)(s) = \sum_{xy=s} f(x)g(y)$$

$$\sum b_k c_{n-k} = a_n.$$

Then  $R[[S]]$  is a ring.

Example of  $f$ :

$$f = a_{\sqrt{3}} + a_{\sqrt{7}} x^{\sqrt{7}} + a_9 x^9 + a_x x^x + \dots$$

Example: ①  $S = (\mathbb{N} \cup \{0\}, \leq) \Rightarrow R[[S]] = R[[x]]$

② if  $a < b$  if  $a \neq b$ , then  $R[[\mathbb{N} \cup \{0\}, \leq]] = R[x]$

③ if  $S = (\mathbb{Z}, \leq)$ ,  $R[[S]] = R[[x^{-1}, x]]$  Laurent series

④ if  $S = (G, \leq)$ ,  $R[[S]] = R[G]$  group ring

⑤ if  $S = \mathbb{Z}/\mathbb{Z}_n$ ,  $R[[S]] = \frac{R[x]}{\langle x^n - 1 \rangle} \rightarrow$  cyclic codes!

Brook field (2004) :

$R$  : Noetherian

$(M, \leq)$  positive, artinian, narrow, strictly ordered

$M$  finitely generated

$\Rightarrow R[[s]]$  : Noetherian

Generalized  
Hilbert basis  
Theorem.

Nasr (2014) :

$R$  : ACCPL(R)

$R$  : Domain

$\alpha$  : 1-1

$\Rightarrow R[[x, \alpha]], R[x, \alpha] : \text{ACCPL}(R).$

Can we do better than domain?

$\alpha$ -Rigid ring: if  $\alpha(a) = 0$ , then  $a = 0$ .

$\alpha$ -rigid  $\rightarrow R$  is reduced.  
 $\alpha$  is 1-1.

Nasr (2014) :

$R$  : ACCPL(R)

$R$  : ACC on ann

$R$  :  $\alpha$ -rigid

$\Rightarrow R[[x, \alpha]] \text{ ACCPL}(R)$

# Generalized power series (Monoid ring):

$R$ : ring

$(S, \leq)$ : strictly ordered monoid ( $a > b \Rightarrow ac > bc$  or  $ac < bc$ ).

$\omega: S \rightarrow \text{End}(R)$ ,  $\omega(s) = \omega_s \in \text{End}(R)$ .

$$R[[S, \omega]] = \left\{ f: S \rightarrow R \right\}, \text{supp}(f) = \{s \in S; f(s) \neq 0\}$$

all  $a_i \neq 0$  in powerseries

$\text{Supp}(f)$ : artinian  $s_1 > s_2 > s_3 > \dots$  terminates

narrow  $\hookrightarrow$  each pairwise in comparable subset has finitely element.

and

$$(fg)(s) = \sum_{xy=s} f(x) \omega_x(g(y))$$

all  $\sum b_k \alpha^k (c_{n-k}) = a_n$ .

Then  $R[[S, \omega]]$  is a ring.

Example:

$$f = 6x^\pi - 7x^{\sqrt{2}} + 9x^3 + \dots$$

$S$ -rigid: if  $a \omega_s(a) = 0$ , then  $a = 0$ .

Liu (2004):

$$\left. \begin{array}{l} R \text{ domain} \\ S, R: \text{Accp} \end{array} \right\} \implies R[[S]]; \text{Accp}$$

$S$ : strictly ordered

Ziembowski, Mazurek (2009):

$$\left. \begin{array}{l} R: \text{Accpl}(R) \\ \omega_s: \text{injective} \\ R: \text{Domain} \end{array} \right\} \implies R[[S, \omega]]: \text{Accpl}(R)$$

domain

Case (R):  $\omega_s$  preserve nonunits of R

Can we do better than domain?

Padashnik, M:

$$\left. \begin{array}{l} R: \text{Accpl}(R) \\ R: \text{Acc on ann} \\ R: S\text{-rigid} \end{array} \right\} \implies R[[S, \omega]]: \text{Accpl}(R)$$

reduced.

$S$ : artinian-narrow

$S$ -finite .  $S \subseteq R$  multiplicative subset .

$I \triangleleft R$  :  $S$ -finite  $\iff \exists$  finitely-generated  $J \subseteq R$   
s.t.  $\exists s \in S, Is \subseteq J \subseteq I$

$S$ -Noetherian.

All ideals of  $R$  are  $S$ -finite.

Archimedean ring.

For nonunit element  $a \in R$ ,  $\bigcap_{n=1}^{\infty} a^n R = 0$ .

Note:  
ACCP  $\rightsquigarrow \bigcap_{n=1}^{\infty} a_1 \dots a_n R = 0$ ,  $a_i$  : nonunit

$\begin{cases} \text{ACCP} \not\Rightarrow \text{Archimedean} \\ \text{Archimedean} \not\Rightarrow \text{ACCP} \end{cases}$

ACCP domain  $\implies$  Archimedean domain.

$\sigma, S$ -anti-archimidean.

$\exists s \in S \subseteq R$  multiplicative s.t.

$$\bigcap_{\substack{n=1 \\ k_i \geq 0}}^{\infty} R \sigma^{k_1}(s) \sigma^{k_2}(s) \sigma^{k_3}(s) \dots \sigma^{k_n}(s) \neq \langle 0 \rangle.$$

Padashnik, M :

$\sigma$  : 1-1, onto

$S \subseteq R$

$R$  :  $\sigma, S$ -anti-archimidean

$R$  :  $S$ -Noetherian

$\implies R[x; \sigma] : S$  Noetherian  
weak  $\curvearrowright$   $S$ -Hilbert-basis? 😊

Theorem.

$R$  : domain

$$\bigcap_{n=1}^{\infty} Ra^n = 0$$

$\implies Q(R[x; \sigma])$  has  $\infty$  transcendental degree.

Padashnik (2019):

$R$ : domain

$(S, \leq)$  strictly ordered

$R$ : Archimedean

$\omega_s$ : 1-1



$R[[S, \omega]]$

Archimedean  
domain

For  $S$ -rigid case,

Padashnik, M, Qureshi (2020)

$R$ :  $\alpha$ -rigid

$R$ : Archimedean

$R$ : ACC on ann



$R[[X, \alpha]]$

Archimedean  
reduced.

Pf steps:  $R[[X, \alpha]] = A$

Let  $\Gamma = \{g \mid \cap A g^n \neq 0\}$ ,  $I_f = \langle \theta(fg) \rangle$ .

$T = \{ \text{Ann}(\bigcup_{i \in \mathbb{N}} I_{c_i g}) \mid c_i \in R, g \in \Gamma \}$

$\rightarrow T \neq \emptyset \iff T$  has maximal element

$V = \text{Ann}(\bigcup I_{c_i g})$ ,

Step 1.  $V$  is a two-sided completely prime ideal.

Step 2.  $\bar{\alpha}^{-1}(V) = V$

Step 3.  $W = \frac{R}{V}[[X, \bar{\alpha}]]$  Arch. domain

Step 4.  $\bar{f}$  is nonunit in  $W$ .

Step 5.  $A$  is reduced Archimedean domain

THANK

YOU